

RUELLE OPERATOR THEOREM FOR NONEXPANSIVE SYSTEMS

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Abstract

The Ruelle operator theorem has been studied extensively both in dynamical systems and iterated function systems. In this paper we study the Ruelle operator theorem for nonexpansive systems. Our theorems give some sufficient conditions for the Ruelle operator theorem to be held for a nonexpansive system.

1. Introduction

Ruelle introduced a convergence theorem to study the equilibrium state of an infinite one-dimensional lattice gas in his famous paper [22]. Bowen [3] further set up the theorem as the convergence of powers of a Ruelle operator on the space of continuous functions on a symbolic space. More precisely, let

$$\Sigma = \{1, \dots, N\}^{\mathbb{N}} = \{\omega = i_0 i_1 \dots i_{n-1} \dots \mid i_{n-1} \in \{1, \dots, N\}, n = 1, 2, \dots\}$$

be the one-sided symbolic space and

$$\sigma : \omega = i_0 i_1 \dots i_{n-1} \dots \rightarrow \sigma(\omega) = i_1 \dots i_{n-1} \dots$$

be the left shift of Σ . Then (Σ, σ) is called a symbolic system. Let ϕ be a Hölder continuous function on Σ (a potential). Let $C(\Sigma)$ be the space of all continuous functions on Σ . The *Ruelle operator* is defined as

$$\mathcal{T}f(x) = \sum_{y \in \sigma^{-1}(x)} e^{\phi(y)} f(y), \quad f \in C(\Sigma). \quad (1.1)$$

It is a positive operator, that is, $\mathcal{T}f > 0$ whenever $f > 0$.

Let ρ be the spectral radius of the operator

$$\mathcal{T} : C(\Sigma) \rightarrow C(\Sigma).$$

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It is known that ϱ is the unique positive simple maximal eigenvalue of \mathcal{T} acting on the space of all Hölder continuous functions on Σ (see, for example, [12]). It was then proved that \mathcal{T} has a unique positive eigenfunction $h \in C(\Sigma)$ and a unique probability eigenmeasure $\mu \in C^*(\Sigma)$ corresponding to the eigenvalue $\varrho > 0$ (see, for example, [3]). And moreover, for any $f \in C(\Sigma)$, $\varrho^{-n}\mathcal{T}^n(f)$ converges uniformly to a constant multiple of h . This is called the Ruelle operator theorem. In this theorem, $\sigma : \Sigma \rightarrow \Sigma$ is an expanding dynamical system. More general results about the Ruelle operator theorem for expanding dynamical systems and contractive iterated function systems (IFS) have been also obtained. We give a partial list in the literature [5, 6, 7, 8, 25, 26].

Recently a parabolic system has drawn a great attention to people who are interested in the Ruelle operator theorem (refer to [1, 16, 17, 21, 24, 27, 28, 29, 30]). However, in this case, it is known that the bounded eigenfunction of the spectral radius ϱ of T may not exist [14], and even if the eigenfunction exists, ϱ may not be an isolated point of the spectrum [2]. So far the results known are far from satisfactory. And a study of such a system remains a challenge problem. Lau and Ye studied the Ruelle operator theorem for a nonexpansive system in a recent paper [15]. In this paper we continue to study the above mentioned problem for a nonexpansive system. In the paper [15], one requirement is that one of the iterations of the IFS must be strictly contractive. It is important to remove this requirement because many examples of IFS will not satisfy this requirement. In this paper, we remove this requirement. It is a major improvement.

Our iterated function system (IFS) $\{w_j\}_{j=1}^m$ in this paper is weakly contractive as defined by

$$\alpha_{w_j}(t) := \sup_{|x-y| \leq t} |w_j(x) - w_j(y)| < t, \quad \forall t > 0, \quad 1 \leq j \leq m$$

or, more generally, nonexpansive as defined by

$$|w_j(x) - w_j(y)| \leq |x - y|, \quad 1 \leq j \leq m.$$

For the weakly contractive case, the invariant compact set K exists as in the contractive case (Hata [9]). For the nonexpansive case we can take the smallest compact invariant K (see Proposition 2.1 for the additional assumption). With each w_j , we associate a positive continuous function p_j as a weight function (or potential function). We can set up the Ruelle operator as in (1.2) on the space $C(K)$ of continuous functions on K ,

$$T(f)(x) = \sum_{j=1}^m p_j(x) f(w_j(x)), \quad f \in C(K). \quad (1.2)$$

Let ϱ still be the spectral radius of the operator

$$T : C(K) \rightarrow C(K).$$

Definition 1.1. We call $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ a *nonexpansive system*, if all maps w_j are nonexpansive and all potentials $p_j(x)$ are Dini continuous on X .

The main result in this paper which we are particularly interested in is that

Theorem 1.2 (Main Theorem). Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a nonexpansive system. Suppose

$$\sup_{x \in K} \sum_{j=1}^m p_j(x) \sup_{y \neq x} \frac{|w_j(x) - w_j(y)|}{|x - y|} < \varrho.$$

Then the Ruelle operator theorem holds for this nonexpansive system.

We will prove a more general result (Theorem 4.5) in §4. Actually, the above theorem is a special case of this more general result. The results in this paper extend the results in [15]. However, as we pointed out before, it is a non-trivial generalization: In the paper [15], one of the iterations of the IFS must be strictly contractive and this is removed in this paper. It is an important improvement. Therefore, we provide a Ruelle operator theorem for a system to which each branch contains an indifferent fixed point (see Remark 4.6 and Example 4.7 in the end of this paper).

In practice, it is difficult to calculate the spectral radius ϱ of T . But since T is a positive operator, we have that $\|T^n 1\| = \|T^n\|$ and

$$\varrho = \lim_n \|T^n\|^{\frac{1}{n}} = \lim_n \|T^n 1\|^{\frac{1}{n}}.$$

Therefore, from the formula of $T^n 1$ (see the formula before Proposition 2.3 in §2), a simple but useful lower bound of ϱ is

$$\min_{x \in K} \sum_{j=1}^m p_j(x) \leq \varrho. \quad (1.3)$$

If we replace the ϱ by $\min_{x \in K} \sum_{j=1}^m p_j(x)$ in the above theorem, we can have a simple checkable sufficient condition.

Corollary 1.3. Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a nonexpansive system. If

$$\sup_{x \in K} \sum_{j=1}^m p_j(x) \cdot \sup_{y \neq x} \frac{|w_j(x) - w_j(y)|}{|x - y|} < \min_{x \in K} \sum_{j=1}^m p_j(x),$$

then the Ruelle operator theorem holds for this nonexpansive system.

It is obvious that if $\{w_j\}_{j=1}^m$ is a contractive IFS, then the conditions in the above theorem and the above corollary and Theorem 4.5 latter are trivially satisfied. The condition of the above theorem is similar to the *average contractive* condition of Barnsley *et al* [2] where they assumed that

$\sum_{j=1}^m p_j(x) = 1$, hence $\varrho = 1$. It is also similar to the one given by Hennion [10], but he considered the case that each p_j is a Lipschitz continuous function on X . Regarding T as defined on the Lipschitz continuous space, he showed that the essential spectral radius $\varrho_e(T)$ is strictly less than the spectral radius $\varrho(T)$, and then the Ruelle operator theorem holds. Furthermore, a general formula for the essential spectral radius $\varrho_e(T)$ for a general C^α IFS or Zygmund IFS can be found in [1]. Using this formula, one can check whether the essential spectral radius $\varrho_e(T)$ is strictly less than the spectral radius $\varrho(T)$, and then check the Ruelle operator theorem. However, these methods do not work for the weakly contractive (or, more generally, nonexpansive) case. The reason is that, in this case, $\varrho(T)$ is not an isolated point of the spectrum, and $\varrho(T) = \varrho_e(T)$ (refer to [20, 23]). Note that [19, 13] contain some results showing that $\varrho(T) = \varrho_e(T)$ is held under some weaker smoothness assumptions (for example, Dini continuity) even in the contractive case. Therefore, the result in this paper provides a new method to check the Ruelle operator theorem for some weakly contractive (or, more generally, nonexpansive) IFS.

We would like to note that most people study an IFS on some Euclidean space. This is because the existence of a compact invariant subset K for a contractive or a weakly contractive IFS needs the structure of a Euclidean space (see [11, 9]). However, arguments in the proofs of this paper only need to assume that K is a compact Hausdorff metric space, in particular, when we studies a dynamical system $\sigma : K \rightarrow K$ defined on a compact Hausdorff metric space K satisfying certain Markov property. More precisely, $K = \cup_{j=1}^m K_j$ is the union of finitely many pairwise disjoint compact subsets $\{K_j\}_{j=1}^m$ such that each $\sigma : K_j \rightarrow K$ is a homeomorphism. Then let w_j be the inverse of $\sigma : K_j \rightarrow K$ for each $1 \leq j \leq m$ and define $(K, \{w_j\}_{j=1}^m)$. It can be thought as an IFS as well. Our results in this paper are true for such a nonexpansive IFS $(K, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$.

The paper is organized as follows. In §2, we will present some elementary facts about the Ruelle operator and prove Proposition 2.1. We will introduce the Ruelle operator theorem in §3 and set up the basic criteria for the assertion of the Ruelle operator theorem. We will prove our main result in §4.

2. Preliminaries

Consider the system

$$(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m),$$

where $X \subseteq \mathbb{R}^d$ is a compact subset, $w_j : X \rightarrow X$, $1 \leq j \leq m$, are continuous maps and the $p_j(x)$, $1 \leq j \leq m$, are positive functions on X (they are called

weights or potentials associated with w_j). We say that a map $w : X \rightarrow X$ is *nonexpansive* if

$$|w(x) - w(y)| \leq |x - y|, \quad \forall x, y \in X;$$

weakly contractive if

$$\alpha_w(t) := \sup_{|x-y| \leq t} |w(x) - w(y)| < t, \quad \forall t > 0.$$

It is clear that contractivity implies weak contractivity which also implies nonexpansiveness. A simple nontrivial example of a weakly contractive map is $w(x) = x/(1+x)$ on $[0, 1]$. We call

$$(X, \{w_j\}_{j=1}^m)$$

a weakly contractive IFS if all w_j , $1 \leq j \leq m$, are weakly contractive; a nonexpansive IFS if all w_j , $1 \leq j \leq m$, are nonexpansive.

A function $p(x)$ defined on X is called Dini continuous if

$$\int_0^1 \frac{\alpha_p(t)}{t} dt < \infty$$

where

$$\alpha_p(t) = \sup_{|x-y| \leq t} |p(x) - p(y)|.$$

For any $0 < \theta < 1$, we consider the following summation

$$S_{\theta,p} = \sum_{n=0}^{\infty} \alpha_p(\theta^n a)$$

where a is the diameter of X . Then, $p(x)$ is Dini continuous is equivalent to saying that $S_{\theta,p}$ is summable, that is,

$$S_{\theta,p} < \infty.$$

Throughout the paper, we always assume the potentials p_j 's are positive Dini continuous functions on X . If $\{w_j\}_{j=1}^m$ is a contractive IFS with the contractive constant $0 < \tau < 1$, that is,

$$\sup_{x \neq y \in X} \frac{|w_j(x) - w_j(y)|}{|x - y|} \leq \tau,$$

then the Dini condition on all p_j can be replaced by the summable condition

$$\max_{1 \leq j \leq m} S_{\tau,p_j} < \infty.$$

However, if $\{w_j\}_{j=1}^m$ is a nonexpansive IFS, we will not have such a constant $0 < \theta < 1$. Thus the Dini condition on potentials is different from the summable condition on potentials. The methods presented before (see e.g. [1, 5, 7, 8, 10, 15, 16, 17, 21, 25, 26, 27]) do not work for the system considered in this paper. We need to find a more sharp method to prove the Ruelle operator theorem under our sufficient conditions.

Definition 2.1. Let $p_j, 1 \leq j \leq m$, be positive Dini continuous functions on X . We call

$$(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m),$$

a nonexpansive (or weakly contractive) system, if the IFS $(X, \{w_j\}_{j=1}^m)$ is nonexpansive (or weakly contractive).

Hata studied the invariant sets of the weakly contractive IFS on $X \subseteq \mathbb{R}^d$ in [9]. By using the existence of fixed points for the weakly contractive maps, he showed the existence of a unique nonempty compact $K \subseteq X$ invariant under $\{w_j\}_{j=1}^m$, i.e.

$$K = \bigcup_{j=1}^m w_j(K).$$

For $J = (j_1 j_2 \cdots j_n)$, $1 \leq j_i \leq m$, let

$$w_J(x) = w_{j_1} \circ w_{j_2} \circ \cdots \circ w_{j_n}(x).$$

Then

$$\lim_{|J| \rightarrow \infty} |w_J(K)| = 0$$

and

$$K = \bigcap_{n=1}^{\infty} \bigcup_{|J|=n} w_J(K).$$

However, for a general IFS, an invariant set may not be unique. However, we have

Proposition 2.2. Suppose $\{w_j\}_{j=1}^m$ is a nonexpansive IFS on the compact subset X with at least one w_j being weakly contractive. Then there exists a unique smallest nonempty compact set K such that

$$K = \bigcup_{j=1}^m w_j(K).$$

Moreover for any $x \in K$, the closure of $\{w_J(x) : |J| = n, n \in \mathbb{N}\}$ is K , i.e.

$$\overline{\{w_J(x) : |J| = n, n \in \mathbb{N}\}} = K.$$

Proof. Let

$$\mathcal{F} = \{F \mid \bigcup_{j=1}^m w_j(F) \subseteq F\}.$$

By using the standard Zorn's lemma argument, there exists a minimal compact subset K such that

$$K = \bigcup_{j=1}^m w_j(K).$$

To show that such K is unique, we assume without loss of generality that w_1 is weakly contractive. If $J_n = (1 \cdots 1)$ (n -times), then $\lim_{n \rightarrow \infty} |w_{J_n}(X)| = 0$.

Let K' be another minimal compact invariant set and let $x \in K$ and $y \in K'$. Then

$$\lim_{n \rightarrow \infty} w_{J_n}(x) = \lim_{n \rightarrow \infty} w_{J_n}(y) \in K \cap K'.$$

Hence

$$K \cap K' \neq \emptyset,$$

and $w_j(K \cap K') \subseteq K \cap K'$. From the minimality of K , we conclude that $K = K'$, and deduce the last statement of the proposition. \square

Throughout the paper we will consider either weakly contractive IFS or the IFS in Proposition 2.2. Hence the set K is uniquely defined. Furthermore, we can assume without loss of generality that the diameter

$$|K| = \sup\{|x - y| : x, y \in K\} = 1.$$

Let $C(K)$ be the space of all continuous functions on K . For such an system, we define an operator $T : C(K) \rightarrow C(K)$ by

$$Tf(x) = \sum_{j=1}^m p_j(x) f(w_j(x)).$$

We call T the *Ruelle operator* associated to the nonexpansive system

$$(K, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m).$$

The dual operator T^* on the measure space $M(K)$ is given by

$$T^*\mu(E) = \sum_{j=1}^m \int_{w_j^{-1}(E)} p_j(x) d\mu(x) \quad \text{for any Borel set } E \subseteq K$$

(see e.g. [2]).

For $J = (j_1 j_2 \cdots j_n)$, $1 \leq j_i \leq m$, define

$$w_J = w_{j_1} \circ w_{j_2} \circ \cdots \circ w_{j_n}$$

and

$$p_{w_J}(x) = p_{j_1}(w_{j_2} \circ w_{j_3} \circ \cdots \circ w_{j_n}(x)) \cdots p_{j_{n-1}}(w_{j_n}(x)) p_{j_n}(x).$$

Then

$$T^n f(x) = \sum_{|J|=n} p_{w_J}(x) f(w_J x).$$

Let $\varrho = \varrho(T)$ be the spectral radius of T . Since T is a positive operator, we have that $\|T^n 1\| = \|T^n\|$ and

$$\varrho = \lim_n \|T^n\|^{\frac{1}{n}} = \lim_n \|T^n 1\|^{\frac{1}{n}}.$$

Proposition 2.3. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a nonexpansive system with at least one weakly contractive w_j . Let T be the Ruelle operator on $C(K)$. Then*

$$(i) \min_{x \in K} \varrho^{-n} T^n 1(x) \leq 1 \leq \max_{x \in K} \varrho^{-n} T^n 1(x) \text{ for all } n > 0;$$

- (ii) if there exist $\lambda > 0$ and $0 < h \in C(K)$ such that $Th = \lambda h$, then $\lambda = \varrho$ and there exist $A, B > 0$ such that

$$A \leq \varrho^{-n} T^n 1(x) \leq B \quad \forall n > 0.$$

Proof. We will prove the second inequality of (i), the first inequality is similar. Suppose it is not true, then there exists an integer k such that $\|T^k 1\| < \varrho^k$. Hence

$$\varrho = (\varrho(T^k))^{\frac{1}{k}} \leq \|T^k\|^{\frac{1}{k}} = \|T^k 1\|^{\frac{1}{k}} < \varrho,$$

which is a contradiction. To prove the second assertion we let $a_1 = \min_{x \in K} h(x)$, $a_2 = \max_{x \in K} h(x)$. Then

$$0 < \frac{a_1}{a_2} \leq \frac{h(x)}{a_2} = \frac{\lambda^{-n}}{a_2} T^n h(x) \leq \lambda^{-n} T^n 1(x) = \lambda^{-n} \|T^n\|.$$

Similarly we can show that $\lambda^{-n} \|T^n\| \leq a_2/a_1$. Hence $\varrho = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} = \lambda$. \square

We call the operator $T : C(K) \rightarrow C(K)$ *irreducible* (see [15]) if for any non-trivial, non-negative $f \in C(K)$ and for any $x \in K$, there exists an integer $n > 0$ such that $T^n f(x) > 0$.

Proposition 2.4. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a nonexpansive system with at least one weakly contractive w_j . Then the Ruelle operator T is irreducible and*

$$\dim\{h \in C(K) : Th = \varrho h, h \geq 0\} \leq 1.$$

If $h \geq 0$ is a ϱ -eigenfunction of T , then $h > 0$.

Proof. The proof can be found in [15]. We include the details here for the sake of completeness. For any given $f \in C(K)$ with $f \geq 0$ and $f \not\equiv 0$, let $V = \{x \in K : f(x) > 0\}$. For any $x \in K$, by Proposition 2.3, there exists a multi-index J_0 such that $w_{J_0}(x) \in V$. Let $n_0 = |J_0|$, then

$$T^{n_0} f(x) = \sum_{|J|=n_0} p_{w_J}(x) f(w_J x) \geq p_{w_{J_0}}(x) f(w_{J_0} x) > 0.$$

This proves that T is irreducible.

For the dimension of the eigensubspace, we suppose that there exist two independent strictly positive ϱ -eigenfunctions $h_1, h_2 \in C(K)$. Without loss of generality we assume that $0 < h_1 \leq h_2$ and $h_1(x_0) = h_2(x_0)$ for some $x_0 \in K$. Then $h = h_2 - h_1 (\geq 0)$ is a ϱ -eigenfunction of T and $h(x_0) = 0$. It follows that $T^n h(x_0) = \varrho^n h(x_0) = 0$, which contradicts to the irreducibility of T . Hence the dimension of the ϱ -eigensubspace is at most 1.

The strict positivity of h follows directly from the irreducibility of T . \square

3. Ruelle Operator Theorem

Proposition 3.1. *Let ϱ_e be the essential spectral radius of T . Suppose $\varrho_e < \varrho$. Then there exists a $h \in C(K)$ with $h > 0$, a probability measure $\mu \in M(K)$ and a constant $0 < b < 1$ such that for any $f \in C(K)$,*

$$\|\varrho^{-n}T^n f - \langle \mu, f \rangle h\|_\infty = O(b^n).$$

Proof. Without loss of generality, we assume that

$$\max_{x \in K} \sum_{j=1}^m p_j(x) \leq 1.$$

Then, we can prove, by induction, that

$$\sup_{n>0} \|T^n 1\| = \sup_{n>0} \max_{x \in K} \sum_{|J|=n} p_{w_J}(x) \leq 1.$$

Then, the operators sequence $n^{-1}T^n$ converges weakly to 0. Note that (see [18] or [1])

$$\varrho_e = \lim_{n \rightarrow \infty} \left(\inf \{ \|T^n - Q\| : Q \text{ is compact on } C(K) \} \right)^{\frac{1}{n}}.$$

From this, together with the assumption $\varrho_e < \varrho$ and theorem VIII.8.7 in [4], it follows that T is quasi-compact [10]. By making use of Hennion's method [10], we can deduce the assertion. \square

In the following, we are interested in the case that $\varrho_e = \varrho$. We first give a basic criterion for the existence of the eigenfunction corresponding to the spectral radius ϱ in this case.

Proposition 3.2. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a nonexpansive system with at least one weakly contractive w_j . Suppose*

- (i) *there exist $A, B > 0$ such that $A \leq \varrho^{-n}T^n 1(x) \leq B$ for any $x \in K$ and $n > 0$, and*
- (ii) *for any $f \in C(K)$, $\{\varrho^{-n}T^n f\}_{n=1}^\infty$ is an equicontinuous sequence.*

Then there exists a unique positive function $h \in C(K)$ and a unique probability measure $\mu \in M(K)$ such that

$$Th = \varrho h, \quad T^* \mu = \varrho \mu, \quad \langle \mu, h \rangle = 1.$$

*Moreover, for every $f \in C(K)$, $\varrho^{-n}T^n f$ converges to $\langle \mu, f \rangle h$ in the supremum norm, and for every $\xi \in M(K)$, $\varrho^{-n}T^{*n} \xi$ converges weakly to $\langle \xi, h \rangle \mu$.*

Proof. The proof can be found in [15], and we omit it. \square

Definition 3.3. Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a nonexpansive system. We say that the Ruelle operator theorem holds for this system if there exists a unique positive function $h \in C(K)$ and a unique probability $\mu \in M(K)$ such that

$$Th = \varrho h, \quad T^* \mu = \varrho \mu, \quad \langle \mu, h \rangle = 1,$$

and for every $f \in C(K)$, $\varrho^{-n} T^n f$ converges to $\langle \mu, f \rangle h$ in the supremum norm.

In the next section, we will study the Ruelle operator theorem for a nonexpansive system under the framework in Proposition 3.2.

4. Some sufficient conditions

Throughout this section we consider a nonexpansive system $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$. And, we assume the nonexpansive IFS $(X, \{w_j\}_{j=1}^m)$ containing at least one weakly contractive w_j . We will prove the Ruelle operator theorem by applying Proposition 3.2.

In the next lemma we will see that the Dini condition on all p_j also implies a similar nature property of the “bounded distortion property”. Recall that an equivalent condition for a function $p(x)$ on K to be Dini continuous is

$$\sum_{n=0}^{\infty} \alpha_p(\theta^n) < \infty$$

for any $0 < \theta < 1$.

Lemma 4.1. Suppose $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ is a nonexpansive system. Let

$$\alpha(t) = \max_{1 \leq j \leq m} \{\alpha_{\log p_j}(t)\}.$$

Let $0 < \theta < 1$ and let

$$a = \sum_{n=0}^{\infty} \alpha(\theta^n).$$

For any fixed $x, y \in K$, if $J = (j_1 \cdots j_n)$ satisfies the condition:

$$|w_{j_{i+1}} \circ \cdots \circ w_{j_n}(x) - w_{j_{i+1}} \circ \cdots \circ w_{j_n}(y)| \leq \theta^{n-i} \quad \forall 1 \leq i < n.$$

Then

$$p_{w_J}(x) \leq e^a p_{w_J}(y).$$

Proof. The inequality follows from the estimate that

$$\begin{aligned} \left| \log \frac{p_{w_J}(x)}{p_{w_J}(y)} \right| &\leq \sum_{i=1}^n |\log p_{j_i}(w_{j_{i+1}} \circ \cdots \circ w_{j_n}(x)) - \log p_{j_i}(w_{j_{i+1}} \circ \cdots \circ w_{j_n}(y))| \\ &\leq \sum_{i=1}^n \alpha(\theta^{n-i}) \leq a. \end{aligned}$$

□

Proposition 4.2. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a nonexpansive system. Suppose*

- (i) $r := \sup_{x \in K} \min_{1 \leq j \leq m} \sup_{y \neq x} \frac{|w_j(x) - w_j(y)|}{|x - y|} < 1$;
- (ii) *there exist constants $A, B > 0$ such that $A \leq \varrho^{-n} T^n 1(x) \leq B$ for any $x \in K$ and $n > 0$.*

Then the Ruelle operator theorem holds for this IFS.

We would like to point out that the condition (i) of Proposition 4.2 is a generalization of the condition (i) of [Theorem 4.2, 15]. We extend theorem 4.2 of [15] so that the system considered in this paper satisfies the condition (i) of Proposition 4.2.

Proof. The proof is the same as the one of [Theorem 4.2, 15], and we omit it. \square

For any integer n , we let $I^n = \{J = (j_1 j_2 \cdots j_n) : 1 \leq j_i \leq m\}$, and let

$$D_n = \{(n_1, n_2, \dots, n_k) : 0 < n_i < n_{i+1} \text{ and } n_k \leq n\} \cup \{(0)\}.$$

For any $J \in I^n$ and any $0 \leq k < l \leq n$, we define $J|_l^k = (j_{n-l+1} j_{n-l+2} \cdots j_{n-k})$. We let $J|_l^k = \emptyset$ if $k = l$.

For any multi-index J and $x \in K$, we let

$$\gamma_J(x) = \sup_{y \neq x} \frac{|w_J(x) - w_J(y)|}{|x - y|}.$$

For convenience, we let $\gamma_J(x) = 1$ and $p_{w_J}(x) = 1$ if $|J| = 0$.

Proposition 4.3. *Let $\{D(k)\}_{k=1}^\ell$ be a partition of I^n , and let*

$$0 = n_0^{(k)} < n_1^{(k)} < \cdots < n_{t_k}^{(k)} = n \quad \forall 1 \leq k \leq \ell. \quad (4.1)$$

Then for any $x \in K$,

$$\sum_{k=1}^{\ell} \sum_{J \in D(k)} p_{w_J}(x) \cdot \prod_{t=1}^{t_k} \gamma_{J|_{n_t}^{n_{t-1}^{(k)}}} (w_{J|_{n_{t-1}^{(k)}}}^0 x) \leq a^n,$$

provided that

$$\sup_{x \in K} \sum_{j=1}^m p_j(x) \cdot \gamma_j(x) \leq a. \quad (4.2)$$

Proof. Note the fact that

$$p_{w_J}(x) = \prod_{i=0}^{n-1} p_{j_{n-i}}(w_{J|_i^0} x) \quad \forall J = (j_1 j_2 \cdots j_n).$$

From (4.2), we can deduce inductively that for any integer n ,

$$\sum_{|J|=n} p_{w_J}(x) \cdot \prod_{i=0}^{n-1} \gamma_{J|_{i+1}^i}(w_{J|_i^0} x) \leq a^n. \quad (4.3)$$

For any multi-index $J = (j_1 j_2 \cdots j_N)$ and $x \in K$, we have

$$\frac{|w_J(x) - w_J(y)|}{|x - y|} = \prod_{i=0}^{N-1} \frac{|w_{j_{N-i}}(w_{J|_i^0} x) - w_{j_{N-i}}(w_{J|_i^0} y)|}{|w_{J|_i^0}(x) - w_{J|_i^0}(y)|}, \quad \forall y \neq x.$$

This implies that

$$\gamma_J(x) \leq \prod_{i=0}^{N-1} \gamma_{j_{N-i}}(w_{J|_i^0} x). \quad (4.4)$$

From the assumption (4.1), using the same argument as (4.4), we deduce that for any J with $|J| = n$,

$$\prod_{t=1}^{t_k} \gamma_{J|_{n_t^{(k)}}^{n_{t-1}^{(k)}}}(w_{J|_{n_t^{(k)}}^0} x) \leq \prod_{i=0}^{n-1} \gamma_{J|_{i+1}^i}(w_{J|_i^0} x). \quad (4.5)$$

Note that $\{D(k)\}_{k=1}^\ell$ is a partition of $I^n (= \{J : |J| = n\})$. We have

$$\begin{aligned} & \sum_{k=1}^\ell \sum_{J \in D(k)} p_{w_J}(x) \cdot \prod_{t=1}^{t_k} \gamma_{J|_{n_t^{(k)}}^{n_{t-1}^{(k)}}}(w_{J|_{n_t^{(k)}}^0} x) \\ & \leq \sum_{|J|=n} p_{w_J}(x) \cdot \prod_{i=0}^{n-1} \gamma_{J|_{i+1}^i}(w_{J|_i^0} x) \quad (\text{by (4.5)}) \\ & \leq a^n \quad (\text{by (4.3)}). \end{aligned}$$

Thus, the conclusion follows. \square

As a consequence of Proposition 4.2, we have

Proposition 4.4. *Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a nonexpansive system. Suppose that*

(i) *there exists k such that*

$$\sup_{x \in K} \sum_{|J|=k} p_{w_J}(x) \cdot \gamma_J(x) < \varrho^k;$$

(ii) *there exist constants $A, B > 0$ such that $A \leq \varrho^{-n} T^n 1(x) \leq B$ for any $x \in K$ and $n > 0$.*

Then the Ruelle operator theorem holds.

Proof. By (i) there exists a $0 < \eta < 1$ such that

$$\sup_{x \in K} \sum_{|J|=k} p_{w_J}(x) \cdot \gamma_J(x) \leq \eta \varrho^k$$

This, together with Proposition 4.3, implies that for any $x \in K$ and $\ell \in \mathbb{N}$,

$$\sum_{|J|=\ell k} p_{w_J}(x) \cdot \prod_{t=1}^{\ell} \gamma_{J|_{tk}}^{(t-1)k}(w_{J|_{(t-1)k}^0} x) \leq \eta^\ell \varrho^{\ell k}.$$

By using the argument similar to (4.4), we can prove that for any multi-index J with $|J| = \ell k$,

$$\gamma_J(x) \leq \prod_{t=1}^{\ell} \gamma_{J|_{tk}}^{(t-1)k}(w_{J|_{(t-1)k}^0} x).$$

It follows that

$$\sum_{|J|=\ell k} p_{w_J}(x) \cdot \gamma_J(x) \leq \eta^\ell \varrho^{\ell k}. \quad (4.6)$$

We claim that

$$\sup_{x \in K} \inf_{\ell \in \mathbb{N}} \min_{|J|=\ell k} \gamma_J(x) = 0.$$

Otherwise, we suppose that

$$\sup_{x \in K} \inf_{\ell \in \mathbb{N}} \min_{|J|=\ell k} \gamma_J(x) > 0.$$

Then, there exists a $b_0 > 0$ and a $x_0 \in K$ such that

$$\inf_{\ell \in \mathbb{N}} \min_{|J|=\ell k} \gamma_J(x_0) \geq b_0.$$

This, combined with (4.6) and (ii), implies that for any $\ell \in \mathbb{N}$,

$$\begin{aligned} \eta^\ell &\geq \varrho^{-\ell k} \sum_{|J|=\ell k} p_{w_J}(x_0) \cdot \gamma_J(x_0) \geq b_0 \cdot \varrho^{-\ell k} \sum_{|J|=\ell k} p_{w_J}(x_0) \\ &= b_0 \cdot \varrho^{-\ell k} T^{\ell k} 1(x_0) \geq b_0 A. \quad (\text{by (ii)}) \end{aligned}$$

This contradicts to the choice of $0 < \eta < 1$. Then, the claim follows. And thus, there exists a $\ell_0 \in \mathbb{N}$ and a J_0 with $|J_0| = \ell_0 k$ such that $\sup_{x \in K} \gamma_{J_0}(x) < 1$. Hence, by Proposition 4.2, the Ruelle operator theorem for $T^{\ell_0 k}$ holds. This implies that the Ruelle operator theorem for T holds. \square

Theorem 4.5. Suppose $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ is a nonexpansive system. If there exists k such that

$$\sup_{x \in K} \sum_{|J|=k} p_{w_J}(x) \cdot \gamma_J(x) < \varrho^k, \quad (4.7)$$

then the Ruelle operator theorem holds.

Proof. Since the Ruelle operator theorem for T^k implies the Ruelle operator theorem for T , we may assume $k = 1$ in the hypothesis, so that (4.7) is reduced to

$$\sup_{x \in K} \sum_{j=1}^m p_j(x) \cdot \gamma_j(x) < \varrho. \quad (4.8)$$

This means that the condition (i) of Proposition 4.4 is satisfied. Hence, we need only to show that condition (ii) of Proposition 4.4 is also satisfied, i.e. there exist $A, B > 0$ such that

$$A \leq \varrho^{-n} \sum_{|J|=n} p_{w_J}(x) \leq B \quad \forall n.$$

By (4.8) we can find $0 < \eta < 1$ such that

$$\sup_{x \in K} \sum_{j=1}^m p_j(x) \cdot \gamma_j(x) \leq \eta \varrho. \quad (4.9)$$

For any fixed $x \in K$, choose θ such that $0 < \eta < \theta < 1$. For any integer n and $J \in I^n$, let n_1 be the largest integer such that

$$\gamma_{J|_{n_1}^0}(x) \geq \theta^{n_1},$$

and let $n_2(> n_1)$ be the largest integer such that

$$\gamma_{J|_{n_2}^{n_1}}(w_{J|_{n_1}^0} x) \geq \theta^{n_2-n_1},$$

and so on. Then, we find a sequence $\{n_i\}_{i=1}^{t_J}$ such that

$$\gamma_{J|_{n_{i+1}}^{n_i}}(w_{J|_{n_i}^0} x) \geq \theta^{n_{i+1}-n_i} \quad \forall 1 \leq i \leq n_{t_J} - 1,$$

and

$$\gamma_{J|_i^{n_{t_J}}}(w_{J|_{n_{t_J}}^0}(x)) < \theta^{i-n_{t_J}} \quad \forall n_{t_J} < i \leq n. \quad (4.10)$$

Define $\sigma : I^n \rightarrow D_n$ by

$$\sigma(J) = (n_1, n_2, \dots, n_{t_J}).$$

Then $\#\sigma(I^n) < \infty$. Denote $\sigma(I^n) = \{A_k\}_{k=1}^\ell$, where $A_k \in D_n$. Let

$$D(k) = \{J : \sigma(J) = A_k\}, \quad \forall 1 \leq k \leq \ell.$$

It is clear that

$$D(i) \cap D(j) = \emptyset, \quad \forall i \neq j.$$

Hence, $\{D(k)\}_{k=1}^\ell$ is a partition of I^n .

For any $1 \leq k \leq \ell$, let $A_k = (n_1^{(k)}, n_2^{(k)}, \dots, n_{t_{k-1}}^{(k)})$. For convenience, we let $n_0^{(k)} = 0$ and let $n_{t_k}^{(k)} = n$. By making use of (4.9), it follows from

Proposition 4.3 that

$$S_0 := \sum_{k=1}^{\ell} \sum_{J \in D(k)} p_{w_J}(x) \cdot \prod_{t=1}^{t_k} \gamma_{J|_{n_t^{(k)}}}^{n_{t-1}^{(k)}} (w_J|_{n_{t-1}^{(k)}}^0 x) \leq (\eta\varrho)^n. \quad (4.11)$$

Let

$$\begin{aligned} \Omega(n, k) &= \{J : |J| = n \text{ and } n_{t_J} = k\}, \quad 1 \leq k \leq n, \\ \Omega(n, 0) &= \{J : |J| = n \text{ and } n_{t_J} = 0\}. \end{aligned}$$

Then

$$I^n = \bigcup_{k=0}^n \Omega(n, k).$$

Without loss of generality, we assume that $\Omega(n, n) = \{D(k)\}_{k=1}^{\ell_0}$, where $\ell_0 \leq \ell$. And we let

$$S_1 := \sum_{k=1}^{\ell_0} \sum_{J \in D(k)} p_{w_J}(x) \cdot \prod_{t=1}^{t_k} \gamma_{J|_{n_t^{(k)}}}^{n_{t-1}^{(k)}} (w_J|_{n_{t-1}^{(k)}}^0 x).$$

For any $1 \leq k \leq \ell_0$ and any $J \in D(k)$, we have $n_{t_k-1}^{(k)} = n_{t_J} = n$, and this implies that

$$\prod_{t=1}^{t_k} \gamma_{J|_{n_t^{(k)}}}^{n_{t-1}^{(k)}} (w_J|_{n_{t-1}^{(k)}}^0 x) \geq \prod_{t=1}^{t_k} \theta^{n_t^{(k)} - n_{t-1}^{(k)}} = \theta^n.$$

From this, we conclude that

$$S_1 \geq \sum_{k=1}^{\ell_0} \sum_{J \in D(k)} p_{w_J}(x) \cdot \theta^n = \sum_{J \in \Omega(n, n)} p_{w_J}(x) \cdot \theta^n.$$

This, combined with (4.11), implies that

$$\sum_{J \in \Omega(n, n)} p_{w_J}(x) \cdot \theta^n \leq S_1 \leq S_0 \leq (\eta\varrho)^n.$$

Thus, it follows that

$$\varrho^{-n} \sum_{J \in \Omega(n, n)} p_{w_J}(x) \leq \left(\frac{\eta}{\theta}\right)^n. \quad (4.12)$$

Remember that

$$\alpha(t) = \max_{1 \leq j \leq m} \alpha_{\log p_j}(t)$$

and

$$a = \sum_{k=0}^{\infty} \alpha(\theta^k).$$

Then a is finite because all the p_i are Dini continuous functions on X . For any $n > 0$, we can make use of Proposition 2.3(i) to find $x_n \in K$ such that

$$\varrho^{-n} \sum_{|J|=n} p_{w_J}(x_n) \leq 1. \quad (4.13)$$

For any $J = (j_1 j_2 \cdots j_n) \in \Omega(n, k)$, we have $J|_k^0 \in \Omega(k, k)$. By using (4.10), we can deduce from Lemma 4.1 that

$$p_{w_{J|_n^k}}(w_{J|_k^0} x) \leq e^a p_{w_{J|_n^k}}(y) \quad \forall y \in K.$$

(We use $|K| = 1$ here.) Hence

$$p_{w_J}(x) = p_{w_{J|_n^k}}(w_{J|_k^0} x) p_{w_{J|_k^0}}(x) \leq e^a p_{w_{J|_n^k}}(x_{n-k}) p_{w_{J|_k^0}}(x). \quad (4.14)$$

It follows that

$$\begin{aligned} \varrho^{-n} \sum_{|J|=n} p_{w_J}(x) &= \varrho^{-n} \sum_{k=0}^n \sum_{J \in \Omega(n, k)} p_{w_J}(x) \\ &\leq \varrho^{-n} \sum_{k=0}^n \sum_{J \in \Omega(n, k)} e^a p_{w_{J|_n^k}}(x_{n-k}) p_{w_{J|_k^0}}(x) \quad (\text{by (4.14)}) \\ &\leq e^a \sum_{k=0}^n \left(\varrho^{-n+k} \sum_{|J'|=n-k} p_{w_{J'}}(x_{n-k}) \right) \left(\varrho^{-k} \sum_{J'' \in \Omega(k, k)} p_{w_{J''}}(x) \right) \\ &\leq e^a \sum_{k=0}^n 1 \cdot \left(\frac{\eta}{\theta} \right)^k \quad (\text{by (4.12), (4.13)}). \end{aligned} \quad (4.15)$$

The last term is bounded by $e^a \sum_{k=0}^{\infty} \left(\frac{\eta}{\theta} \right)^k := B_1$. This concludes the upper bound estimate.

For the lower bound estimation, we note that Proposition 2.3(i) and (4.15) implies that for any $n > 0$, there exists $y_n \in K$ such that

$$1 \leq C_n := \varrho^{-n} \sum_{|J|=n} p_{w_J}(y_n) \leq B_1.$$

For any fixed $x \in K$, we let

$$\alpha_J = \sum_{i=0}^{n-1} \alpha(|w_{J|_i^0}(x) - w_{J|_i^0}(y_n)|).$$

Then, we have

$$p_{w_J}(y_n) \leq p_{w_J}(x) e^{\alpha_J}.$$

By (4.10), we have for any $J \in \Omega(n, k)$,

$$|w_{J|_i^0}(x) - w_{J|_i^0}(y_n)| < \theta^{i-k} \quad \forall k < i \leq n.$$

(We use $|K| = 1$ here.) It follows that

$$\alpha_J \leq a + k\alpha(1) \quad \forall J \in \Omega(n, k).$$

Using the same argument as (4.15), we can deduce that

$$\varrho^{-n} \sum_{J \in \Omega(n,k)} p_{w_J}(y_n) \leq e^a \left(\frac{\eta}{\theta}\right)^k.$$

And then, we have

$$\begin{aligned} \varrho^{-n} \sum_{|J|=n} p_{w_J}(y_n) \alpha_J &= \varrho^{-n} \sum_{k=0}^n \sum_{J \in \Omega(n,k)} p_{w_J}(y_n) \alpha_J \\ &\leq \varrho^{-n} \sum_{k=0}^n (a + k\alpha(1)) \sum_{J \in \Omega(n,k)} p_{w_J}(y_n) \\ &\leq e^a \sum_{k=0}^n (a + k\alpha(1)) \left(\frac{\eta}{\theta}\right)^k \leq B_2, \end{aligned}$$

where $B_2 := e^a \sum_{k=0}^{\infty} (a + k\alpha(1)) \left(\frac{\eta}{\theta}\right)^k$. By the convexity of function e^x , we have

$$\begin{aligned} \varrho^{-n} \sum_{|J|=n} p_{w_J}(x) &\geq \varrho^{-n} \sum_{|J|=n} p_{w_J}(y_n) e^{-\alpha_J} \geq \frac{\varrho^{-n}}{C_n} \sum_{|J|=n} p_{w_J}(y_n) e^{-\alpha_J} \\ &\geq e^{-\frac{1}{C_n} \varrho^{-n} \sum_{|J|=n} p_{w_J}(y_n) \alpha_J} \geq e^{-B_2}. \end{aligned}$$

This completes the proof. \square

Remark 4.6. We note that for any multi-index J and $x \in K$,

$$\gamma_J(x) \leq \sup_{y \neq z} \frac{|w_J(z) - w_J(y)|}{|z - y|}.$$

And then, for any integer n , we have

$$\sum_{|J|=n} p_{w_J}(x) \cdot \gamma_J(x) \leq \sum_{|J|=n} p_{w_J}(x) \cdot \sup_{y \neq z} \frac{|w_J(z) - w_J(y)|}{|z - y|}.$$

Hence, Theorem 4.5 in this paper is a generalization of theorem 4.4 in [15]. However, the following example indicates that this generalization is non-trivial.

Example 4.7. Let $X = [0, 1]$, and let $w_1(x) = x - \frac{x^2}{2}$, $w_2(x) = \frac{1}{2} + \frac{x^2}{2}$. Then $w'_1(\cdot) \geq 0$, $w'_2(\cdot) \geq 0$. And $w_1(0) = 0$, $w_2(1) = 1$; $w'_1(0) = w'_2(1) = 1$ and $w'_1(1) = w'_2(0) = 0$.

In this example, both w_1 and w_2 are not strictly contractive. In fact, 0 is the indifferent fixed point of w_1 ; and 1 is the indifferent fixed point of w_2 . It is easy to see that the IFS $(X, \{w_j\}_{j=1}^2)$ is weakly contractive.

Let p_1 be any positive Dini function (not a Lipschitz function) on X with the inequalities $0 < p_1(\cdot) < 1$. Let

$$\delta = \frac{1}{5} \cdot \min_{x \in X} \{p_1(x), 1 - p_1(x)\} > 0,$$

and let

$$g(x) = \begin{cases} \delta - 2^{-1} + x, & \text{if } 2^{-1} - \delta < x \leq 2^{-1} \\ \delta + 2^{-1} - x, & \text{if } 2^{-1} < x < 2^{-1} + \delta \\ 0, & \text{otherwise.} \end{cases}$$

Define a Dini function p_2 on X by

$$p_2(x) = 1 - p_1(x) + g(x) \quad \forall x \in X.$$

Then

$$1 \leq \sum_{j=1}^2 p_j(x) = 1 + g(x) \leq 1 + \delta.$$

And for any $x \in X$,

$$g(x) - \frac{1}{4}p_1(x) < 0 \quad \text{and} \quad g(x) - \frac{1}{4}p_2(x) < 0. \quad (4.16)$$

Let K be the invariant set of the IFS $\{w_j\}_{j=1}^2$. Define

$$Tf(x) = \sum_{j=1}^2 p_j(x) f \circ w_j(x), \quad \forall f \in C(K).$$

Let ϱ be the spectral radius of the operator T . Then, we have

$$1 \leq \varrho \leq 1 + \delta. \quad (4.17)$$

Note that

$$\begin{aligned} \gamma_1(x) &= \sup_{y \neq x} \frac{|w_1(y) - w_1(x)|}{|y - x|} = \sup_{y \neq x} \frac{|y - x - 2^{-1}y^2 + 2^{-1}x^2|}{|y - x|} \\ &= \sup_{y \neq x} \left(1 - \frac{1}{2}(x + y)\right) = 1 - \frac{x}{2}; \\ \gamma_2(x) &= \sup_{y \neq x} \frac{|w_2(y) - w_2(x)|}{|y - x|} = \sup_{y \neq x} \frac{|2^{-1}y^2 - 2^{-1}x^2|}{|y - x|} = \frac{1}{2}(1 + x). \end{aligned}$$

We have

$$\begin{aligned} \sum_{j=1}^2 p_j(x) \cdot \gamma_j(x) &= p_1(x) \cdot \left(1 - \frac{x}{2}\right) + p_2(x) \cdot \frac{1 + x}{2} \\ &\leq \begin{cases} p_1(x) + \frac{3}{4}p_2(x), & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{3}{4}p_1(x) + p_2(x), & \text{if } \frac{1}{2} < x \leq 1, \end{cases} \\ &\leq \begin{cases} 1 + \left(g(x) - \frac{1}{4}p_2(x)\right), & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1 + \left(g(x) - \frac{1}{4}p_1(x)\right), & \text{if } \frac{1}{2} < x \leq 1, \end{cases} \\ &< 1. \quad (\text{by (4.16)}) \end{aligned}$$

This, together with (4.17), implies that

$$\sup_{x \in X} \sum_{j=1}^2 p_j(x) \cdot \gamma_j(x) < \varrho.$$

And then, Theorem 4.5 implies that the Ruelle operator theorem holds for this weakly contractive system.

Because of the equalities

$$\sup_{y \neq z} \frac{|w_j(y) - w_j(z)|}{|y - z|} = 1 \quad \forall j = 1, 2,$$

and

$$\sup_{x \in X} \sum_{j=1}^2 p_j(x) = \sup_{x \in X} (1 + g(x)) = 1 + \delta,$$

by noting that (4.17), the following inequality:

$$\sup_{x \in X} \sum_{j=1}^2 p_j(x) \cdot 1 < \varrho$$

does not hold. Hence, for this system, the condition of theorem 1.2 in [15] is not satisfied.

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